Lecture 28

Final exam period
Tue Dec 10 8am - 10am Taft 316

<u>Symmetric spaces</u>. (A good comprehensive ref is Helgason, "Diff. geom., Lie Groups and Sym. Spaces.")

(M,g) Riemannian mfld $g \in \Gamma(\Sigma^2(T^*M))$

$\forall p \in M$, g_p pos def quad form on TM. $g_p(v,w)$ or $\langle v,w\rangle$

$d(p,q) = \inf\limits_{\gamma\ p \text{ to } q} \int_a^b \sqrt{g(\gamma',\gamma')}\,dt$ Met space inducing mfld top.

Isometry: $f: M \to M$ preserving d $\overset{\text{Myers-Steenrod, 1939}}{\Longleftrightarrow}$ f is smooth and pres g.
($\forall p$ $df_p: (T_pM, g_p) \to (T_{f(p)}M, g_{f(p)})$ is an isom of inner prod)

<u>Prop</u>. Spse M connected, f isom, $f(p)=p$, $df_p = Id$.
Then $f = Id_M$.

<u>Thm</u> (Myers-Steenrod) $\mathrm{Isom}(M,g) = \{f: M \to M \text{ isom of } g\}$ has a unique Lie grp structure that induces the compact-open topology. The action map $\mathrm{Isom}(M,g) \times M \to M$ is smooth.

$\mathrm{Stab}_{\mathrm{Isom}(M)}(p)$ is compact by the prop: it injects into $O(T_pM, g_p)$

With that prep: Let (M,g) be connected from now on.

(M,g) is <u>symmetric</u> if $\forall p \in M$ $\exists$ isom $s_p: M \to M$ s.t.
$s_p(p) = p$ and $(ds_p)_p = -Id_{T_pM}$.

That is, the map $-Id$ on T_pM extends to an isometry of M (necessarily <u>unique</u>, as any two such differ by isom that fixes p and T_pM, so apply prop)

<u>Thm</u> (see Helgason): Let (M,g) be symmetric. Then

1) M is complete (as a metric space)
2) $G := \mathrm{Isom}(M)$ acts transitively, so M is diffeo to the homogeneous space G/K where $K = \mathrm{Stab}_G(p_0)$, $p_0 \in M$.
3) The Riem metric of $M \cong G/K$ is induced by a unique K-invt inner product on $\mathfrak{g}/\mathfrak{k}$. $\quad (TM \cong (G \times \mathfrak{g}/\mathfrak{k})/K)$

So symmetric spaces are homog spaces w/ homog metrics. Which (G,K) arise?

WLOG G connected. (take $\mathrm{Isom}^0(M)$ instead)

Recall $G = \mathrm{Isom}(M)$ $K = \mathrm{Stab}_G(p)$. Consider $\sigma: G \to G$
$\sigma(g) = s_p g s_p$. It is an automorphism and $\sigma^2 = id$. $G^\sigma = \mathrm{fix}(\sigma)$
Also $(G^\sigma)^0 \subset K \subset G^\sigma$.

Such (G,K,σ) is called a <u>Riemannian symmetric pair</u>. (sic)
These are classified.

<u>Thm</u> (de Rham). M a Riem sym space. Then M is isometric to a product $M_+ \times M_0 \times M_-$ where each factor is a Riem sym space and:

1. M_+ is compact, has assoc RSP (G_+, K_+, σ_+), G_+ semisimple/$\mathbb{R}$.
2. M_0 is isometric to $\mathbb{R}^k$ w/ Euclidean metric
3. M_- has assoc RSP (G_-, K_-, σ_-), G_- semisimple, K_- max cpt.

Furthermore M_+, M_0, M_- unique up to isom. $\quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^\perp$

For G ss noncpt and K max cpt, $\exists!$ Riem sym pair str (G,K,σ)

Thus one considers cases 1, 2, 3 separately to understand symm spaces, typically. That is, spse two of the factors are a point

M of type in ① is called compact type

M of type in ② is called Euclidean type (examples clear)

M of type in ③ is called noncompact type.

Ex. S^n w/ induced met is $SO(n+1)/SO(n)$

$\mathbb{CP}^n$ w/ Fubini-Study is $SU(n+1)/SU(n)$

Ex. $\mathbb{H}^n = \left(\mathbb{B}^n, \frac{4\,ds^2_{euc}}{(1-r^2)^2}\right)$ is $SO(n,1)/SO(n)$

$\mathrm{Inner}_0(\mathbb{R}^n) = \left\{ \begin{array}{l} \text{pos def inner products} \\ \text{on } \mathbb{R}^n \text{ s.t. unit cube} \\ \text{has volume } 1 \end{array} \right\} = SL_n\mathbb{R}/SO(n)$

$\mathrm{Herm}_0(\mathbb{C}^n) = \left\{ \begin{array}{l} \text{pos def hermitian} \\ \text{forms on } \mathbb{C}^n \text{ s.t.} \\ \Delta^n \text{ has volume } \pi^n \end{array} \right\} = SL_n\mathbb{C}/SU(n)$

Now G complex semisimple$/\mathbb{C}$ $K \subset G$ max compact (def over $\mathbb{R}$)

$\Rightarrow$ ss$/\mathbb{R}$

Then $X = G/K$ and the Riem str corresp to a K-invt inner prod on $\mathfrak{g}/\mathfrak{k}$.

Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p} = i\mathfrak{k}$ Note

$[\,,\,]$	$\mathfrak{k}$	$\mathfrak{p}$
$\mathfrak{k}$	$\mathfrak{k}$	$\mathfrak{p}$
$\mathfrak{p}$	$\mathfrak{p}$	$\mathfrak{k}$

K-invt inner product on $\mathfrak{p}$ $\langle v,w\rangle = \langle Ad_k v, Ad_k w\rangle$

$\Leftrightarrow$ $\mathfrak{k}$-invt inner product on $\mathfrak{p}$.

K conn $\quad \langle [x,v], w\rangle_{\mathfrak{p}} + \langle v, [x,w]\rangle_{\mathfrak{p}} = 0$.

$\Leftrightarrow$ $\mathfrak{k}$-invt inner prod on $\mathfrak{k}$

If $v,w \in \mathfrak{k}$, let $\langle v,w\rangle_{\mathfrak{k}} = \langle iv, iw\rangle_{\mathfrak{p}}$

Now if $\mathfrak{g}$ simple ($\Rightarrow \mathfrak{k}$ simple), unique up to scale. Killing form.

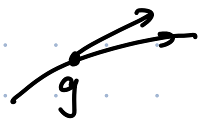
Well, Killing form is <u>negative def</u>, so define for $v,w \in \mathfrak{k}$

$$\langle v,w\rangle = -B(v,w) \qquad B = \text{Killing form of } \mathfrak{k} = \mathrm{tr}(\mathrm{ad}_v \circ \mathrm{ad}_w)$$

Eg $SL_n\mathbb{C} = G \quad SU(n) = K. \quad G/K = \text{herm metrics}$

$$\mathfrak{g} = \text{traceless} \quad \mathfrak{k} = \{X \mid X + \overline{X}^t = 0\} \quad \mathfrak{p} = \{X \mid X - \overline{X}^t = 0\}$$

$$gK \longleftrightarrow h_g(z,w) = \bar{z}^t \bar{g}^t g w$$



tangent vector to G/K? $\quad (g + \varepsilon \dot{g})K$. or better $g(I + \varepsilon i v)K$

We can always take $v \in \mathfrak{k}$

Then if $v,w \in \mathfrak{k}$ rep tgt vecs at same gK this way,

$$\langle v,w\rangle = -\mathrm{Re}\,\mathrm{tr}(vw) \qquad v,w \text{ are antisymmetric}$$

(Because up to scale the Killing form on $SU(n)$ is

$X,Y \longmapsto \mathrm{Re}(\mathrm{tr}(XY).)$ $\quad X,Y$ imag diag show it's neg def.)